Maximum of Dyson Brownian motion and non-colliding systems with a boundary

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Abstract

We prove an equality-in-law relating the maximum of GUE Dyson's Brownian motion and the non-colliding systems with a wall. This generalizes the well known relation between the maximum of a Brownian motion and a reflected Brownian motion.

1 Introduction and Results

Dyson's Brownian motion model of GUE (Gaussian unitary ensemble) is a stochastic process of positions of m particles, $X(t) = (X_1(t), \dots, X_m(t))$ described by the stochastic differential equation,

$$dX_i = dB_i + \sum_{\substack{1\leq j\leq m\\ j\neq i}} \frac{dt}{X_i - X_j}, \quad 1 \leq i \leq m, \tag{1.1}$$

where $B_i, 1 \leq i \leq m$ are independent one dimensional Brownian motions [5]. The process satisfies $X_1(t) < X_2(t) < \cdots < X_m(t)$ for all $t > 0$. We remark that the process X can be started from the origin, i.e., one can take $X_i(0) = 0, 1 \leq i \leq m$. See [8].

One can introduce similar non-colliding system of m particles with a wall at the origin [6, 7, 14]. The dynamics of the positions of the m particles $X^{(C)} = (X_1^{(C)}, \dots, X_m^{(C)})$ satisfying

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$0 < X_1(t) < X_2(t) < \cdots < X_m(t)$ for all $t > 0$ are described by the stochastic differential equation,

$$dX_i^{(C)} = dB_i + \frac{dt}{X_i^{(C)}} + \sum_{\substack{1\le j\le m\\ j\ne i}} \left(\frac{1}{X_i^{(C)} - X_j^{(C)}} + \frac{1}{X_i^{(C)} + X_j^{(C)}} \right) dt,\ 1 \le i \le m. \tag{1.2}$$

This process is referred to as Dyson's Brownian motion of type C. It can be interpreted as a system of m Brownian particles conditioned to never collide with each other or the wall.

One can also consider the case where the wall above is replaced by a reflecting wall [7]. The dynamics of the positions of the m particles $X^{(D)} = (X_1^{(D)}, \ldots, X_m^{(D)})$ satisfying $0 \le X_1(t) < X_2(t) < \cdots < X_m(t)$ for all $t > 0$, is described by the stochastic differential equation,

$$dX_i^{(D)} = dB_i + \frac{1}{2}\mathbf{1}_{(i=1)} dL(t) + \sum_{\substack{1\le j\le m\\ j\ne i}} \left(\frac{1}{X_i^{(D)} - X_j^{(D)}} + \frac{1}{X_i^{(D)} + X_j^{(D)}} \right) dt,\ 1 \le i \le m, \tag{1.3}$$

where $L(t)$ denotes the local time of $X_1^{(D)}$ at the origin. This process will be referred to as Dyson's Brownian motion of type D. Some authors consider a process defined by the s.d.e.s (1.3) without the local time term. In this case the first component of the process is not constrained to remain non-negative, and the process takes values in the Weyl chamber of type D, $\{|x_1| < x_2 < x_3 \ldots < x_m\}$. The process we consider with a reflecting wall is obtained from this by replacing the first component with its absolute value, with the local time term appearing as a consequence of Tanaka's formula.

It is known the processes $X^{(C,D)}$ can be obtained using the Doob h-transform, see [6]. Let $(P_t^{0,(C,D)}; t \ge 0)$ be the transition semigroup for m independent Brownian motions killed on exiting $\{0 < x_1 < x_2 \ldots < x_m\}$, resp. the transition semigroup for m independent Brownian motions reflected at the origin killed on exiting $\{0 \le x_1 < x_2 \ldots < x_m\}$. From the Karlin-McGregor formula, the corresponding densities can be written as

$$\det\{\phi_t(x_i - x'_j) - \phi_t(x_i + x'_j)\}_{1\le i,j\le m}, \tag{1.4}$$

resp.,

$$\det\{\phi_t(x_i - x'_j) + \phi_t(x_i + x'_j)\}_{1\le i,j\le m}, \tag{1.5}$$

where $\phi_t(z) = \frac{1}{\sqrt{2\pi t}} e^{-z^2/(2t)}$. Let

$$\begin{aligned} h^{(C)}(x) &= \prod_{i=1}^m x_i \prod_{1\le i<j\le m} (x_j^2 - x_i^2), \\ h^{(D)}(x) &= \prod_{1\le i<j\le m} (x_j^2 - x_i^2). \end{aligned} \tag{1.6}$$

For notational simplicity we suppress the index C, D for the semigroups and in h in the following. Then one can show that $h(x)$ is invariant for the P_t^0 semigroup and we may define a Markov semigroup by

$$P_t(x, dx') = h(x')P_t^0(x, dx')/h(x). \tag{1.7}$$

This is the semigroup of the Dyson non-colliding system of Brownian motions of type C and D. Similarly to the X process, the processes $X^{(C)}$ and $X^{(D)}$ can also be started from the origin (see [9] or use Lemma 4 in [7] and apply the same arguments as in [8]).

In GUE Dyson's Brownian motion of n particles, let us take the initial conditions to be $X_i(0) = 0, 1 \le i \le n$. The quantity we are interested in is the maximum of the position of the top particle for a finite duration of time, $\max_{0\le s\le t} X_n(s)$. In the sequel we write sup instead of max to conform with common usage in the literature. Let m be the integer such that $n = 2m$ when n is even and $n = 2m-1$ when n is odd. Consider the non-colliding systems of $X^{(C)}, X^{(D)}$ of m particles starting from the origin, $X_i^{(C,D)}(0) = 0, 1 \le i \le m$.

Our main result of this note is

Theorem 1. *Let X and $X^{(C)}, X^{(D)}$ start from the origin. Then for each fixed $t \ge 0$, one has*

$$\sup_{0\le s\le t} X_n(s) \stackrel{d}{=} \begin{cases} X_m^{(C)}(t), & \text{for } n = 2m, \\ X_m^{(D)}(t), & \text{for } n = 2m-1. \end{cases} \tag{1.8}$$

To prove the theorem we introduce two more processes Z_j and Y_j. In the Z process, $Z_1 \le Z_2 \le \ldots \le Z_n$, Z_1 is a Brownian motion and Z_{j+1} is reflected by Z_j, $1 \le j \le n-1$. Here the reflection means the Skorokhod construction to push Z_{j+1} up from Z_j. More precisely,

$$\begin{aligned} &Z_1(t) = B_1(t), \\ &Z_j(t) = \sup_{0\le s\le t}(Z_{j-1}(s) + B_j(t) - B_j(s)), \quad 2 \le j \le n, \end{aligned} \tag{1.9}$$

where $B_i, 1 \le i \le n$ are independent Brownian motions, each starting from 0. The process is the same as the process $(X_1^1(t), X_2^2(t), \ldots, X_n^n(t); t \ge 0)$ studied in section 4 of [15]. The representation (1.9) was given earlier in [2]. In the Y process, $0 \le Y_1 \le Y_2 \le \ldots \le Y_n$, the interactions among Y_i's are the same as in the Z process, i.e., Y_{j+1} is reflected by Y_j, $1 \le j \le n-1$, but Y_1 is now a Brownian motion reflected at the origin (again by Skorokhod construction). Similarly to (1.9),

$$\begin{aligned} &Y_1(t) = B_1(t) - \inf_{0\le s\le t} B_1(s) = \sup_{0\le s\le t}(B_1(t) - B_1(s)), \\ &Y_j(t) = \sup_{0\le s\le t}(Y_{j-1}(s) + B_j(t) - B_j(s)), \quad 2 \le j \le n. \end{aligned} \tag{1.10}$$

From the results in [4, 8, 15], we know

$$(X_n(t); t \ge 0) \stackrel{d}{=} (Z_n(t); t \ge 0) \tag{1.11}$$

and hence

$$\sup_{0\le s\le t} X_n(s) \stackrel{d}{=} \sup_{0\le s\le t} Z_n(s). \tag{1.12}$$

In this note we show

Proposition 2. *The following equalities in law hold between processes:*

$$\begin{aligned}(Y_{2m}(t); t\ge 0) &\stackrel{d}{=} (X_m^{(C)}(t); t\ge 0),\\ (Y_{2m-1}(t); t\ge 0) &\stackrel{d}{=} (X_m^{(D)}(t); t\ge 0),\end{aligned} \tag{1.13}$$

$m \in \mathbb{N}$.

The proof of this proposition is given in Section 2. The idea behind it is that the processes $(Y_i)_{i\ge1}$ and $(X_j^{(C,D)})_{j\ge1}$ could be realized on a common probability space consisting of Brownian motions satisfying certain interlacing conditions with a boundary [15,16]. Such a system is expected to appear as a scaling limit of the discrete processes considered in [3,16]. In this enlarged process, the processes $Y_n(t)$ and $X_m^{(C,D)}(t)$ just represent two different ways of looking at the evolution of a specific particle and so the statement of Proposition 2 follows immediately. Justification of such an approach is however quite involved, and we prefer to give a simple independent proof. See also [4] for another representation of $X_m^{(C,D)}$ in terms of independent Brownian motions.

Then to prove (1.8) it is enough to show

Proposition 3. *For each fixed t we have*

$$\sup_{0\le s\le t} Z_n(s) \stackrel{d}{=} Y_n(t). \tag{1.14}$$

This is shown in Section 3. For $n = 1$ case, this is well known from the Skorokhod construction of reflected Brownian motion [10]. The $n > 1$ case can also be understood graphically by reversing time direction and the order of particles. This relation could also be established as a limiting case of the last passage percolation. In fact the identities in our theorem was first anticipated from the consideration of a diffusion scaling limit of the totally asymmetric exclusion process with 2 speeds [1].

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2 Proof of proposition 2

In this section we prove the relation between $X^{(C,D)}$ and Y, (1.13). The following Lemma is a generalization of the Rogers-Pitman criterion [11] for a function of a Markov process to be Markovian.

Lemma 4. *Suppose that $\{X(t) : t \geq 0\}$ is a Markov process with state space E, evolving according to a transition semigroup $(P_t; t \geq 0)$ and with initial distribution μ. Suppose that $\{Y(t) : t \geq 0\}$ is a Markov process with state space F, evolving according to a transition semigroup $(Q_t; t \geq 0)$ and with initial distribution ν. Suppose further that L is a Markov transition kernel from E to F, such that $\mu L = \nu$ and the intertwining $P_t L = L Q_t$ holds. Now let $f : E \to G$ and $g : F \to G$ be maps into a third state space G, and suppose that*

$$L(x, \cdot) \text{ is carried by } \{y \in F : g(y) = f(x)\} \text{ for each } x \in E.$$

Then we have

$$\{f(X(t)) : t \geq 0\} \stackrel{d}{=} \{g(Y(t)) : t \geq 0\},$$

in the sense of finite dimensional distributions.

Proof of Lemma 4. For any bounded function α on G let $\Gamma_1 \alpha$ be the function $\alpha \circ f$ defined on E and let $\Gamma_2 \alpha$ be the function $\alpha \circ g$ defined on F. Then it follows from the condition that $L(x, \cdot)$ is carried by $\{y \in F : g(y) = f(x)\}$ that whenever h is a bounded function defined on F then

$$L(\Gamma_2 \alpha \times h) = \Gamma_1 \alpha \times Lh, \tag{2.1}$$

which is shorthand for $\int L(x, dy) \Gamma_2 \alpha(y) h(y) = \Gamma_1 \alpha \times Lh$. For any bounded test functions $\alpha_0, \alpha_1, \cdots, \alpha_n$ defined on G, and times $0 < t_1 < \cdots < t_n$, we have, using the previous equation and the intertwining relation repeatedly,

$$\begin{aligned}
&\mathbb{E}[\alpha_0(g(Y(0)))\alpha_1(g(Y(t_1)))\dots\alpha_n(g(Y(t_n)))] \\
&= \nu(\Gamma_2\alpha_0 \times Q_{t_1}(\Gamma_2\alpha_1 \times Q_{t_2-t_1}(\cdots(\Gamma_2\alpha_{n-1} \times Q_{t_n-t_{n-1}}\Gamma_2\alpha_n)\cdots))) \\
&= \mu L(\Gamma_2\alpha_0 \times Q_{t_1}(\Gamma_2\alpha_1 \times Q_{t_2-t_1}(\cdots(\Gamma_2\alpha_{n-1} \times Q_{t_n-t_{n-1}}\Gamma_2\alpha_n)\cdots))) \\
&= \mu(\Gamma_1\alpha_0 \times P_{t_1}(\Gamma_1\alpha_1 \times P_{t_2-t_1}(\cdots(\Gamma_1\alpha_{n-1} \times P_{t_n-t_{n-1}}\Gamma_1\alpha_n)\cdots))) \\
&= \mathbb{E}[\alpha_0(f(X(0)))\alpha_1(f(X(t_1)))\dots\alpha_n(f(X(t_n)))]
\end{aligned} \tag{2.2}$$

which proves the equality in law. $\square$

We let $(Y(t) : t \geq 0)$ be the process Y of n reflected Brownian motions with a wall introduced in the previous section. It is clear from the construction (1.10) that the process Y is a time homogeneous Markov process. We denote its transition semigroup by $(Q_t; t \geq 0)$. It turns out that there is an explicit formula for the corresponding densities. Recall $\phi_t(z) = \frac{1}{\sqrt{2\pi t}} e^{-z^2/(2t)}$. Let us define $\phi_t^{(k)}(y) = \frac{d^k}{dy^k}\phi_t(y)$ for $k \geq 0$ and $\phi_t^{(-k)}(y) = (-1)^k \int_y^\infty \frac{(z-y)^{k-1}}{(k-1)!}\phi_t(z)dz$ for $k \geq 1$.

$$
\begin{array}{|ccccccccc}
z_1^1 & & & & & & & & \\
 & z_1^2 & & & & & & & \\
z_1^3 & & z_2^3 & & & & & & \\
 & z_1^4 & & z_2^4 & & & & & \\
z_1^5 & & z_2^5 & & z_3^5 & & & & \\
 & \vdots & & \vdots & & \ddots & & & \\
z_1^n & & z_2^n & & z_3^n & \ldots & z_n^n & &
\end{array}
$$

Figure 1: The set $\mathbb{K}$. The triangle represents the intertwining relations of the variables z and the vertical line on the left indicates $z_1^{2k+1} \geq 0$, see (2.5),(2.6).The set of variables on the bottom line is denoted by $b(z)$ and the one on the upper right line by $e(z)$.

Proposition 5. *The transition densities $q_t(y, y')$ from $y = (y_1, \ldots, y_n)$ at $t = 0$ to $y' = (y'_1, \ldots, y'_n)$ at t of the Y process can be written as*

$$q_t(y, y') = \det\{a_{i,j}(y_i, y'_j)\}_{1 \leq i,j \leq n} \tag{2.3}$$

where $a_{i,j}$ is given by

$$a_{i,j}(y, y') = (-1)^{i-1} \phi_t^{(j-i)}(y + y') + (-1)^{i+j} \phi_t^{(j-i)}(y - y'). \tag{2.4}$$

The same type of formula was first obtained for the totally asymmetric simple exclusion process by Schütz [13]. The formula for the Z process was given as a Proposition 8 in [15], see also [12].

Proof of Proposition 5. For a fixed y', define $G_t(y, t)$ to be (2.3) as a function of y and t. We check that G satisfies (i) the heat equation, (ii) the boundary conditions $\frac{\partial G}{\partial y_1}|_{y_1=0} = 0$, $\frac{\partial G}{\partial y_i}|_{y_i=y_{i-1}} = 0,\ i = 2, 3, \ldots, n$ and (iii) the initial conditions $G(y, t = 0) = \prod_{i=1}^n \delta(y_i - y'_i)$.

(i) holds since $\phi_t^{(k)}(y)$ for each k satisfies the heat equation. (ii) follows from the relations, $\frac{\partial}{\partial y} a_{1j}(y, y')|_{y=0} = \phi_t^{(j)}(y') + (-1)^{j+1} \phi_t^{(j)}(-y') = 0$ and $\frac{\partial}{\partial y} a_{ij}(y, y') = -a_{i-1,j}(y, y')$. For (iii) we notice that the first term in (2.4) goes to zero as $t \to 0$ for $y, y' > 0$ and the statement for the remaining part is shown in Lemma 7 in [15]. $\square$

For $n = 2m$, resp. $n = 2m - 1$ we take $(X(t), t \geq 0)$ to be Dyson Brownian motion of type C, resp. of type D. The transition semigroup $\big(P_t; t \geq 0\big)$ of this process is given by (1.7).

Let $\mathbb{K}$ denote the set with n layers $z = (z^1, z^2, \ldots, z^n)$ where $z^{2k} = (z_1^{2k}, z_2^{2k}, \ldots, z_k^{2k}) \in \mathbb{R}_+^k$, $z^{2k-1} = (z_1^{2k-1}, z_2^{2k-1}, \ldots, z_k^{2k-1}) \in \mathbb{R}_+^k$ and the intertwining relations,

$$z_1^{2k-1} \leq z_1^{2k} \leq z_2^{2k-1} \leq z_2^{2k} \leq \ldots \leq z_k^{2k-1} \leq z_k^{2k} \tag{2.5}$$

and

$$0 \leq z_1^{2k+1} \leq z_1^{2k} \leq z_2^{2k+1} \leq z_2^{2k} \leq \ldots \leq z_k^{2k} \leq z_{k+1}^{2k+1} \tag{2.6}$$

hold (Fig. 1). Let $n = 2m$ or $n = 2m - 1$ for some integer m. We define a kernel L^0 from $E = \{0 \le x_1 \le \ldots \le x_m\}$ to $F = \{0 \le y_1 \le \ldots \le y_n\}$. For $z \in \mathbb{K}$, define $b(z) = z^n = (z_1^n, \ldots, z_m^n) \in E$, $e(z) = (z_1^1, z_1^2, z_2^3, z_2^4, \ldots, z_m^n) \in F$ and $\mathbb{K}(x) = \{z \in \mathbb{K}; b(z) = x \in E\}, \mathbb{K}[y] = \{z \in \mathbb{K}; e(z) = y \in F\}$. The kernel L^0 is defined by

$$L^0 g(x) = \int_F L^0(x, dy) g(y) = \int_{\mathbb{K}(x)} g(e(z)) dz. \tag{2.7}$$

where the integrals are taken with respect to Lebesgue measure but integrations with respect to z on the RHS is for $b(z) = x$ fixed.

The function h defined at (1.6) is equal to the Euclidean volume of $\mathbb{K}(x)$. Consequently we may define L to be the Markov kernel $L(x, dy) = L^0(x, dy)/h(x)$. In the remaining part of this section we show

Proposition 6.

$$LQ_t = P_t L. \tag{2.8}$$

Now if we apply Lemma 4 with $f(x) = x_m$, $g(y) = y_n$ and the initial conditions starting from the origin we obtain (1.13).

Proof of Proposition 6. The kernels $P_t(x, \cdot)$ and $L(x, \cdot)$ are continuous in x. Thus we may consider x in the interior of E, and it is enough to prove

$$(L^0 Q_t)(x, dy) = (P_t^0 L^0)(x, dy). \tag{2.9}$$

From the definition of the kernel L^0, this is equivalent to showing

$$\int_{\mathbb{K}(x)} q_t(e(z), y) dz = \int_{\mathbb{K}[y]} p_t^0(x, b(z)) dz \tag{2.10}$$

where q_t and p^0 are densities corresponding to Q_t and P_t^0. Integrations with respect to z are on the LHS with $b(z) = x$ fixed and on the RHS with $e(z) = y$ fixed.

Let us consider the case where $n = 2m$. Using the determinantal expressions for q_t and p_t^0 we show that both sides of (2.10) are equal to the determinant of size $2m$ whose (i, j) matrix element is $a_{2i,j}(0, y_j)$ for $1 \le i \le m, 1 \le j \le 2m$ and $a_{2m,j}(x_{i-m}, y_j)$ for $m + 1 \le i \le 2m, 1 \le j \le 2m$.

The integrand of the LHS of (2.10) is

$$q_t(e(z), y) = \det\{a_{i,j}(e(z)_i, y_j)\}_{1 \le i,j \le 2m} \tag{2.11}$$

with $b(z) = x$. We perform the integral with respect to $z^1, \ldots, z^{2m-1}$ in this order. After the integral up to $z^{2l-1}, 1 \le l \le m$, we get the determinant of size $2m$ whose (i, j) matrix element is $a_{2i,j}(0, y_j)$ for $1 \le i \le l$, $a_{2l,j}(z_{i-l}^{2l}, y_j)$ for $l + 1 \le i \le 2l$ and $a_{i,j}(e(z)_i, y_j)$ for $2l + 1 \le i \le 2m$. Here we use a property of $a_{i,j}$,

$$a_{i,j}(y, y') = \int_y^\infty a_{i-1,j}(u, y') du, \tag{2.12}$$

and do some row operations in the determinant. The case for $l = m$ gives the desired expression.

The integrand of the RHS of (2.10) is

$$p_t^0(x, z^{2m}) = \det(a_{2m,2m}(x_i, z_j^{2m}))_{1\le i,j\le m} \tag{2.13}$$

with the condition $e(z) = y$. We perform the integrals with respect to $(z_1^{2m}, \ldots, z_{m-1}^{2m}), (z_1^{2m-1}, \ldots, z_{m-1}^{2m-1}), \ldots, z_1^4, z_1^3$ in this order. We use properties of $a_{i,j}$,

$$a_{i,j}(y, y') = -\int_{y'}^{\infty} a_{i,j+1}(y, u)du, \tag{2.14}$$

$$a_{2i,2j}(x, 0) = 0, \quad a_{2i,2i-1}(0, y) = 1, \quad a_{2i,j}(0, y) = 0,\ 2i \le j. \tag{2.15}$$

After each integration corresponding to a layer of $\mathbb{K}$ we simplify the determinant using column operations. We also expand the size of the determinant after an integration corresponding to $(z_1^{2l}, \ldots, z_{l-1}^{2l})$ for $1 \le l \le m$, by adding a new first row

$$\big(\underbrace{1, 1, \ldots, 1}_{l}, \underbrace{0, 0, \ldots, 0}_{2m-2l+1}\big) =$$
$$\big(a_{2l,2l-1}(0, z_1^{2l-1}), \ldots, a_{2l,2l-1}(0, z_l^{2l-1}), a_{2l,2l}(0, e(z)_{2l}), \ldots, a_{2l,2m}(0, e(z)_{2m}))\big) \tag{2.16}$$

together with a new column. After the integrals up to $(z_1^{2l-1}, \ldots, z_{l-1}^{2l-1})$ have been performed, we obtain the determinant of size $2m - l + 1$,

$$\begin{vmatrix} a_{2(l+i-1),2(l-1)}(0, z_j^{2(l-1)}) & a_{2(l+i-1),j+l-1}(0, e(z)_{j+l-1}) \\ a_{2m,2(l-1)}(x_{i-m+l-1}, z_j^{2(l-1)}) & a_{2m,j+l-1}(x_{i-m+l-1}, e(z)_{j+l-1}) \end{vmatrix}. \tag{2.17}$$

Here $1 \le i \le m - l + 1$ (resp. $m - l + 2 \le i \le 2m - l + 1$) for the upper expression (resp. the lower expression) and $1 \le j \le l - 1$ (resp. $l \le j \le 2m - l + 1$) for the left (resp. right) expression. For $l = 1$ this reduces to the same determinant as for the LHS.

The case $n = 2m - 1$ is almost identical. Similar arguments show that both sides of (2.9) are equal to a determinant size $2m - 1$ whose (i, j) matrix element is $a_{2i,j}(0, y_j)$ for $1 \le i \le m - 1, 1 \le j \le 2m - 1$ and $a_{2m-1,j}(x_{i-m+1}, y_j)$ for $m + 1 \le i \le 2m - 1, 1 \le j \le 2m - 1$. $\square$

3 Proof of proposition 3

Using (1.10) repeatedly, one has

$$Y_n(t) = \sup_{0\le t_1\le \ldots \le t_n\le t} \sum_{i=1}^{n} (B_i(t_{i+1}) - B_i(t_i)) \tag{3.1}$$

with $t_{n+1} = t$. By renaming $t - t_{n-i+1}$ by t_i and changing the order of the summation, we have

$$Y_n(t) = \sup_{0 \le t_1 \le \ldots \le t_n \le t} \sum_{i=1}^{n} (B_{n-i+1}(t - t_{i+1}) - B_{n-i+1}(t - t_i)). \tag{3.2}$$

Since $\tilde{B}_i(s) := B_{n-i+1}(t) - B_{n-i+1}(t-s) \stackrel{d}{=} B_i(s)$,

$$Y_n(t) \stackrel{d}{=} \sup_{0 \le t_1 \le \ldots \le t_n \le t} \sum_{i=1}^{n} (B_i(t_i) - B_i(t - t_{i-1})) = \sup_{0 \le s \le t} Z_n(t). \tag{3.3}$$

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